Problem 1

We are asked for final velocity and acceleration for two different initial velocities. Let’s do velocity first. We know the potential energy and the initial velocity, so we can easily solve for the final velocity:

\[ E_i = E_f \]
\[ K_i + U_i = K_f + U_f \]
\[ \frac{1}{2}mv_0^2 + U(x_0) = \frac{1}{2}mv^2 + U(x) \]
\[ \frac{1}{2}mv_0^2 + U(x_0) - U\left(\frac{b}{2}\right) = \frac{1}{2}mv^2 \]

Now, we know \( U(x_0) = 0 \) and

\[ U\left(\frac{b}{2}\right) = U_0 \left(3 \left(\frac{1}{2}\right)^2 - 2 \left(\frac{1}{2}\right)^3\right) = U_0 \left(\frac{3}{4} - \frac{1}{4}\right) = \frac{U_0}{2} \]

so

\[ \frac{1}{2}mv^2 = \frac{1}{2}mv_0^2 + 0 - \frac{U_0}{2} \]
\[ v^2 = v_0^2 - \frac{U_0}{m} \]

substituting the two values for \( v_0 \) gives:

(a) \[ v^2 = \frac{3U_0}{m} - \frac{U_0}{m} \implies v = \sqrt{\frac{2U_0}{m}} \]

(b) \[ v^2 = \frac{U_0}{m} - \frac{U_0}{m} \implies v = 0 \]

Now, for the acceleration: We get the acceleration from \( F = ma \), so it just depends on the potential energy function, not the initial velocity, and so it is the same in both cases:

\[ a = \frac{F}{m} = \frac{1}{m} \left(-\frac{dU}{dx}\right) \]
\[ = -\frac{U_0}{m} \left(6x \frac{-6x^2}{b^2} \right) \]
\[ = -\frac{U_0}{mb} \left(\frac{x}{b} - \left(\frac{x}{b}\right)^2\right) \]

Substituting \( x = \frac{b}{2} \) gives

\[ a = -\frac{U_0}{mb} \left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3U_0}{2mb} \]
Problem 2

It’s just a 1-d collision problem, so we use conservation of momentum:

\[ p_i = p_f \]
\[ 5M v_0 + 0 = 5M \frac{2}{3} v_0 + M v \]
\[ v = \frac{5}{3} v_0 \]

Now, we must check if this collision is elastic. There are two ways to check. One is to compare the initial and final energy: \( E_i = \frac{1}{2} 5M v_0^2 \) and

\[ E_f = \frac{1}{2} 5M \left( \frac{2}{3} v_0 \right)^2 + \frac{1}{2} M \left( \frac{5}{3} v_0 \right)^2 \]
\[ = \frac{1}{2} 5M \frac{4}{9} v_0^2 + \frac{1}{2} M \frac{25}{9} v_0^2 \]
\[ = \frac{1}{2} 5M v_0^2 \]

So the collision is elastic. Can you think of the other way to check?

Problem 3

This is a 2-d collision, so we apply conservation of momentum for each of the two rectangular coordinates: In the \( x \)-direction:

\[ m v_1 \cos \theta - m v_2 \cos \theta = 2m v \cos \phi \]

In the \( y \)-direction:

\[ m v_1 \sin \theta + m v_2 \sin \theta = 2m v \sin \phi \]

Simplifying these equations gives:

\[ v_1 - v_2 = 2v \frac{\cos \phi}{\cos \theta} \]
\[ v_1 + v_2 = 2v \frac{\sin \phi}{\sin \theta} \]

Adding the two equations gives:

\[ v_1 = v \left( \frac{\cos \phi}{\cos \theta} + \frac{\sin \phi}{\sin \theta} \right) \]

Subtracting them gives

\[ v_2 = v \left( \frac{-\cos \phi}{\cos \theta} + \frac{\sin \phi}{\sin \theta} \right) \]
Problem 4

\[
\begin{align*}
\dot{\mathbf{r}} &= \cos \theta \, \dot{x} + \sin \theta \, \dot{y} \\
\dot{\mathbf{r}} &= -\sin \theta \, \dot{x} + \cos \theta \, \dot{y} \\
\frac{d\hat{\mathbf{r}}}{d\theta} &= \dot{\theta} \\
\frac{d\hat{\mathbf{r}}}{dt} &= \dot{\theta} \omega \\
\frac{d\mathbf{r}}{dt} &= \frac{d\mathbf{r}}{dt} + \frac{d\mathbf{r}}{dt} + \dot{\mathbf{r}} + r \dot{\omega} \hat{\theta} \\
\ddot{\mathbf{r}} &= \frac{d^2\mathbf{r}}{dt^2} + \frac{dr}{dt} \frac{d\mathbf{r}}{dt} + \dot{\mathbf{r}} + r \dot{\omega} \hat{\theta} - r \omega^2 \hat{\mathbf{r}} \\
\ddot{\mathbf{r}} &= \left( \frac{d^2r}{dt^2} - r \omega^2 \right) \hat{r} + \left( 2 \omega \frac{dr}{dt} + r \alpha \right) \hat{\theta}
\end{align*}
\]
Problem 5

We need both the radial and tangential components of the acceleration of a point on the outside of the pulley. It will be in circular motion, so according to the result of problem 4 the radial component is \( a_r = -r\omega^2 \) and the tangential component is \( a_\theta = r\alpha \). We can get \( \omega \) and \( \alpha \) from the speed and acceleration of the block: \( \omega = v/R \) and \( \alpha = a/R \).

First, let's find \( a_r = -R\omega^2 \). We want velocity, so we use conservation of energy:

\[
Mgh = \frac{1}{2}Mv^2 + \frac{1}{2}I\omega^2
\]
\[
Mg\frac{R}{3} = \frac{1}{2}M(R\omega)^2 + \frac{1}{2}\left(\frac{1}{2}MR^2\right)\omega^2
\]
\[
Mg\frac{R}{3} = \frac{3}{4}MR^2\omega^2
\]
\[
R\omega^2 = \frac{4}{9}g = a_r
\]

Now, let's find \( a_\theta = R\alpha \). We want acceleration, so we use Newton's law:

Free Body Diagrams look like this:

Newton's Law for the pulley:

\[
\Sigma \tau = TR = I\alpha = \left(\frac{1}{2}MR^2\right)\alpha
\]

and for the block

\[
\Sigma F = Mg - T = Ma = MR\alpha
\]

It's easy to eliminate \( T \): Multiply the second equation through by \( R \) and add to the first.

\[
MgR = \frac{1}{2}MR^2\alpha + MR^2\alpha = \frac{3}{2}MR^2\alpha
\]
\[
R\alpha = \frac{2}{3}g = a_\theta
\]

So, we have both components. Thus:

\[
a = \sqrt{a_r^2 + a_\theta^2}
\]
\[
= \sqrt{\left(\frac{4}{9}g\right)^2 + \left(\frac{2}{3}g\right)^2}
\]
\[
= g\sqrt{\frac{52}{81}}
\]