Problem 1. (I announced in class, on several occasions, that this *exact* problem would be on the exam.)

$$\hat{r} = \cos\theta\,\hat{x} + \sin\theta\,\hat{y} \qquad \hat{\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y}$$

$$\frac{d\hat{r}}{d\theta} = -\sin\theta\,\hat{x} + \cos\theta\,\hat{y} = \hat{\theta} \qquad \frac{d\hat{\theta}}{d\theta} = -\cos\theta\,\hat{x} - \sin\theta\,\hat{y} = -\hat{r}$$

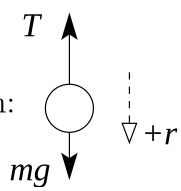
$$\frac{d\hat{r}}{dt} = \frac{d\hat{r}}{d\theta}\frac{d\theta}{dt} = \hat{\theta}\omega \qquad \frac{d\hat{\theta}}{dt} = \frac{d\hat{\theta}}{d\theta}\frac{d\theta}{dt} = -\hat{r}\omega$$

$$\begin{aligned}\vec{v} &= \frac{d\vec{r}}{dt} = \frac{d}{dt}r\,\hat{r} \\ &= \frac{dr}{dt}\hat{r} + r\frac{d\hat{r}}{dt} \\ &= \frac{dr}{dt}\hat{r} + r\omega\,\hat{\theta}\end{aligned}$$

$$\begin{aligned}\vec{a} &= \frac{d\vec{v}}{dt} = \frac{d}{dt}\left(\frac{dr}{dt}\hat{r} + r\omega\,\hat{\theta}\right) \\ &= \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\frac{d\hat{r}}{dt} + \frac{dr}{dt}\omega\,\hat{\theta} + r\frac{d\omega}{dt}\hat{\theta} + r\omega\frac{d\hat{\theta}}{dt} \\ &= \frac{d^2r}{dt^2}\hat{r} + \frac{dr}{dt}\omega\,\hat{\theta} + \frac{dr}{dt}\omega\,\hat{\theta} + r\alpha\,\hat{\theta} - r\omega^2\,\hat{r} \\ &= \underline{\underline{\left(\frac{d^2r}{dt^2} - r\omega^2\right)\hat{r} + \left(2\omega\frac{dr}{dt} + r\alpha\right)\hat{\theta}}}\end{aligned}$$

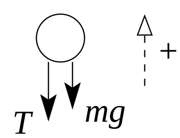
Problem 2. (Very similar to 7.46. See also 7.63 and 5.51.)

Let's draw free-body diagrams and write down Newton's second law for the mass when it is at the bottom and at the top (we let T_t and T_b be the tension at the top and bottom, and let v_t and v_b be the velocity at the top and bottom):

Bottom: 

$$\begin{aligned} mg - T_b &= ma_r \\ &= -m\frac{v_b^2}{l}\end{aligned}$$

$$T_b = mv_b^2/l + mg$$

Top: 

$$\begin{aligned} -mg - T_t &= ma_r \\ &= -m\frac{v_t^2}{l}\end{aligned}$$

$$T_t = mv_t^2/l - mg$$

So, we see have equations relating T_b to v_b and T_t to v_t so we need an equation relating v_b to v_t. The equation that does that is just conservation of energy:

$$E_b = E_t \implies \frac{1}{2}mv_b^2 = \frac{1}{2}mv_t^2 + mg\,(2l) \implies mv_b^2 = mv_t^2 + 4mgl$$

Substituting the above

$$(T_b - mg)l = (T_t + mg)l + 4mgl \implies T_b - mg = T_t + mg + 4mg \implies \underline{\underline{T_t = T_b - 6mg}}$$

Problem 3. (*Pretty much identical* to 9.89.)

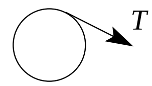
(a) $I = \frac{1}{2}MR^2 + \frac{1}{2}(2M)(2R)^2 = \left(\frac{1}{2} + 4\right)MR^2 = \frac{9}{2}MR^2$

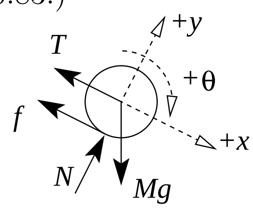
(b)
$$\begin{aligned} E_i &= E_f \\ Mgh &= \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2 \\ &= \frac{1}{2}\frac{9}{2}MR^2\left(\frac{v}{R}\right)^2 + \frac{1}{2}Mv^2 \\ &= \left(\frac{9}{4} + \frac{1}{2}\right)Mv^2 = \frac{11}{4}Mv^2 \\ & \underline{\underline{v = \sqrt{\frac{4gh}{11}}}}\end{aligned}$$

(c)
$$\begin{aligned} E_i &= E_f \\ Mgh &= \frac{1}{2}I\omega^2 + \frac{1}{2}Mv^2 \\ &= \frac{1}{2}\frac{9}{2}MR^2\left(\frac{v}{2R}\right)^2 + \frac{1}{2}Mv^2 \\ &= \left(\frac{9}{16} + \frac{1}{2}\right)Mv^2 = \frac{17}{16}Mv^2 \\ & \underline{\underline{v = \sqrt{\frac{16gh}{17}}}}\end{aligned}$$

Problem 4. (This was kind of a combination of 10.63 and 10.83.)

Free-body diagrams looks like this:





For the pulley: $\Sigma\tau = I\alpha \quad\Longrightarrow\quad TR = \frac{MR^2}{2}\frac{a}{R} \quad\Longrightarrow\quad T = \frac{Ma}{2}$

For the disk: $\Sigma\tau = I\alpha \quad\Longrightarrow\quad fR = \frac{MR^2}{2}\frac{a}{R} \quad\Longrightarrow\quad f = \frac{Ma}{2}$

And also: $\Sigma F_x = Mg\sin\theta - f - T = Ma$

Thus: $Mg\sin\theta - \frac{Ma}{2} - \frac{Ma}{2} = Ma \quad\Longrightarrow\quad a = \underline{\underline{\frac{g\sin\theta}{2}}}$

Problem 5. (See Example 10.13, problem 10.91, and the merry-go-round problem we worked in class.)

$$W = \Delta K = K_f - K_i = \frac{1}{2}I_2\omega_2^2 - \frac{1}{2}I_1\omega_1^2$$

So we need to find ω_2 in terms of the provided quatities. We can do that because there is no external torque, so we know the angular momentum about the rotation axis is constant.

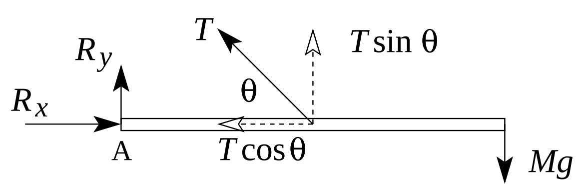
$$L_i = L_f \quad\Longrightarrow\quad I_1\omega_1 = I_2\omega_2 \quad\Longrightarrow\quad \omega_2 = \frac{I_1}{I_2}\omega_1$$

Substituting that into the above we have

$$W = \frac{1}{2}I_2\left(\frac{I_1}{I_2}\omega_1\right)^2 - \frac{1}{2}I_1\omega_1^2 = \underline{\underline{\frac{1}{2}I_1\omega_1^2\left(\frac{I_1}{I_2} - 1\right)}}$$

Problem 6. (This was a *simplified* version of problem 11.14.)

Free-body diagrams looks like this:



(a) Look at the torques about point A:

$\Sigma\tau_A = 0 \quad\Longrightarrow\quad T\sin\theta\,\frac{L}{2} - Mg\,L = 0 \quad\Longrightarrow\quad \underline{\underline{T = \frac{2Mg}{\sin\theta}}}$

(b) Look at the Forces:

$\Sigma F_x = R_x - T\cos\theta = 0 \quad\Longrightarrow\quad R_x = T\cos\theta = \underline{\underline{\frac{2Mg}{\tan\theta}}}$

$\Sigma F_y = R_y + T\sin\theta - Mg = 0 \quad\Longrightarrow\quad R_y = Mg - 2Mg = -Mg \quad\Longrightarrow\quad \underline{\underline{|R_y| = Mg}}$ and so it is <u>pointing down</u>.